Hamiltonian formulation of the problem of a single vortex evolution on a beta-plane

G.K. Korotaev

Marine Hydrophysical Institute, Russian Academy of Sciences, Sevastopol, Russian Federation
e-mail: korotaevgren@mail.ru

Evolution of a single intense vortex on the ß-plane is studied based on the "elliptical approach" which was introduced by Legras and Dritschel for description of the eddies with uniform vorticity. A vortex is represented by two round patches of uniform vorticity – one inside another. One of them constitutes the vortex core and another – the trap zone. Radiation of the Rossby waves is not considered in this study. Application of the "elliptical approach" permits to generalize the earlier-proposed theory of intense vortex evolution on the ß-plane. Besides the Rossby and Zhukovsky–Kutta forces, it includes the inertia force in the equations describing the vortex motion. It is shown that the deduced system of equations is written down using non-canonical variables; and it can be represented in the generalized Hamiltonian form in case the vortex motion equations are supplemented with the equation of absolute vorticity conservation. Being analyzed, the deduced equations' solutions show that they both provide new interpretation of the vortex self-propagation on the ß-plane and permit to characterize high-frequency oscillations of the vortex center position. Thus the represented theory permits to explain similar oscillations sometimes arising in the numerical experiments.

Keywords: intensive vortex, ß-plane, Hamiltonian formalism.

DOI: 10.22449/1573-160X-2016-6-15-23

Introduction. Discovery of synoptic eddies in the ocean and construction of the ocean circulation eddy-resolving models have stimulated research on evolution of intensive vortices on the ß-plane. Analytic and numerical studies have shown that, being affected by planetary vorticity, a single cyclonic vortex on the ß-plane moves to the northwest. It was demonstrated in [1, 2] that in course of its motion the vortex had formed an almost circular trap zone within which the liquid particles had been involved in a rotary motion around its center. For a cyclonic vortex, the liquid particles inside the trap zone are of almost constant negative vorticity. Due to the method of contour dynamics applied in [3], evolution of the vortex which initially had a form of a circular spot with constant vorticity was rather accurately numerically calculated. After a while, when the trap zone is formed, there arises a configuration which can be roughly described as a circular spot of positive vorticity placed inside another circular spot with negative vorticity. According to the general theses of the theory represented in [1, 2], the total vorticity of such a configuration should slightly exceed zero.

Numerical simulations of long-term evolution of a single synoptic vortex on the ß-plane also show (see, for example, the figures from [3]) that the higher-frequency oscillations of its center in the vicinity of the motion general trajectory are imposed on the vortex smooth motion. At certain parameters of the vortex, these oscillations are accompanied by deformation of its almost circular shape that is explained by the vortex unstable configuration. If the vorex parameters are far from critical, almost circular shape of the vortex core and the trap zone remains in course of its movement in spite of high-frequency oscillations of the vortex center position. The vortex simple configuration permits to use the developed in [4] so-called "elliptical approach" for a more detailed, as compared to the represented in [1, 2], description of the synoptic vortex evolution on the *ß*-plane. It will be shown below that application of the "elliptical approach" permits to write down the equations describing the vortex zone transfer and relatively high-frequency oscillations of its center in the Hamiltonian form.

The second section of the paper contains discussion of the proposed model; and the equations describing the vortex evolution are deduced. It is shown in the third section that the derived equations are of the non-canonical Hamiltonian form. In the fourth section, solutions of the obtained equations are analyzed. Section Conclusion represents discussion of the solution physical features.

Model of a single vortex evolution. Let us consider the following mo-del of a vortex on the *ß*-plane. At the initial moment of time the vortex is assumed to be of a constant positive vorticity ω_2 within the circle with radius R_2. It is shown in [1] and confirmed in [2, 3] by numerical simulations that after the vortex displaced along the meridian, it constituted almost a circular core with positive vorticity surrounded by the trap zone with negative vorticity and was of almost a circular shape with radius $R_{1.}$ Based on the arguments in [1], the relative vorticity within the trap zone is assumed to be uniform and equal toω_1. Since there are no reasons to suppose that the centers of the trap zone and the vortex core coincide, we assume that their coordinates are X_1, Y_1 and X_2, $Y_{2,}$ respectively (axis X is directed to the east, axis Y – to the north). The *ß*-effect impact will be taken into account only within the trap zone. Outside the trap zone, the liquid motion is assumed to be potential that is, according to [1], permissible for the intense vortex (similar scheme of allowing for the *ß*-effect within the framework of the "elliptical approach" was proposed by Bernard Legras in a private discussion). The vortex-radiated Rossby waves also will not be taken into account, whereas interaction between the vortex and the trap zone will be considered. Under such conditions, the stream function can be represented as the following sum $\Psi = \Psi_1 + \Psi_2$ where each of the terms is derived from the following equations:

$$\nabla^2\psi_1 = \begin{cases} \omega_1 - \beta(y - Y_1),\ r \le R_1, \\ 0,\ r > R_1, \end{cases} \tag{1}$$

$$\nabla^2\psi_2 = \begin{cases} \omega_2,\ r' \le R_2, \\ 0,\ r' > R_2, \end{cases} \tag{2}$$

where $r^2 = (x - X_1)^2 + (y - Y_1)^2$, $r'^2 = (x - X_2)^2 + (y - Y_2)^2$.

Equations (1) and (2) are easily solved as follows:

$$\psi_1 = \begin{cases} \frac{1}{4}\omega_1(r^2 - R_1^2) - \frac{\beta}{8}r^2(y - Y_1) + \frac{\beta R_1^2}{4}(y - Y_1),\ r \le R_1, \\ \frac{\omega_1 R_1^2}{2}\ln\left(\frac{r}{R_1}\right) + \frac{\beta R_1^4}{8r^2}(y - Y_1),\ r > R_1, \end{cases} \tag{3}$$

$$\psi_2 = \begin{cases} \frac{1}{4}\omega_2 (r'^2 - R_2^2),\ r' \le R_2, \\ \frac{\omega_2 R_2^2}{2} \ln\left(\frac{r'}{R_2}\right),\ r' > R_2. \end{cases} \tag{4}$$

Using the idea of [4] let us write out the equations describing translation of the vortex core and the trap zone boundaries

$$S_i \frac{dX_i}{dt} = \oint_{\Gamma_i} \psi dx \text{ and } S_i \frac{dY_i}{dt} = \oint_{\Gamma_i} \psi dy, \tag{5}$$

where Γ_1 and S_1 are the trap zone boundary and square; Γ_2 and S_2 are the vortex core boundary and square. Having substituted (3) and (4) to (5) we obtain the following system of equations:

$$\begin{aligned} \frac{dX_1}{dt} &= \frac{\omega_2 R_2^2}{2R_1^2}(Y_2 - Y_1) - \frac{\beta R_1^2}{8}, \\ \frac{dY_1}{dt} &= -\frac{\omega_2 R_2^2}{2R_1^2}(X_2 - X_1), \end{aligned} \tag{6}$$

$$\begin{aligned} \frac{dX_2}{dt} &= -\frac{1}{2}\omega_1 (Y_2 - Y_1) - \frac{\beta R_1^2}{4} + \frac{\beta R_2^2}{8} + \frac{\beta}{8}\left[(X_2 - X_1)^2 + 3(Y_2 - Y_1)^2\right], \\ \frac{dY_2}{dt} &= \frac{1}{2}\omega_1 (X_2 - X_1) - \frac{\beta}{4}(X_2 - X_1)(Y_2 - Y_1). \end{aligned} \tag{7}$$

The system of equations (6) and (7) is closed by the equation of absolute vorticity conservation within the trap zone

$$\omega_1 + \beta Y_1 = \text{const}. \tag{8}$$

Or, having taken into account (6) we obtain

$$\frac{d\omega_1}{dt} = \beta \frac{\omega_2 R_2^2}{2R_1^2}(X_2 - X_1). \tag{9}$$

Hamiltonian formulation of the problem. Equations (6), (7) and (9) are derived under the assumption that both the vortex core and the trap zone are of a circular form. Approximately circular form of the vortex core and the trap zone will not be, apparently, violated only if the displacement of the core and trap zone centers is substantially smaller than the trap zone radius. Therefore, it is advisable to omit the quadratic terms in the right-hand side of equation (7) as the ones whose order of magnitude is smaller. At that the system of equations (6), (7) and (9) takes on the Hamiltonian form with the Hamiltonian

$$\begin{aligned} H(X_1, Y_1, X_2, Y_2, \omega_1) &= \pi R_1^2 \left\{ \left[\frac{\omega_2 R_2^2}{2R_1^2}(Y_2 - Y_1) - \frac{\beta R_1^2}{8} \right]^2 + \left(\frac{\omega_2 R_2^2}{2R_1^2} \right)^2 (X_2 - X_1)^2 \right\} + \\ &+ \frac{\pi R_1^2 R_2^2}{4}\omega_1\omega_2 - \frac{\pi R_2^4}{8}\omega_1\omega_2 + \frac{\pi R_1^4}{16}\omega_1^2. \end{aligned} \tag{10}$$

However, it should be noted that the system of equations (6), (7) and (9) is written using the non-canonical coordinates and the integral (8) is the Casimir. The corresponding simplex matrix for the system of equations (6), (7) and (9) is as follows:

$$J = \begin{bmatrix} 0 & -\frac{1}{\pi\omega_2 R_2^2} & 0 & 0 & 0 \\ \frac{1}{\pi\omega_2 R_2^2} & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & -\frac{\omega_1 R_1^2}{\pi\omega_2^2 R_2^4} & -\frac{\beta}{\pi\omega_2 R_2^2} \\ 0 & 0 & \frac{\omega_1 R_1^2}{\pi\omega_2^2 R_2^4} & 0 & 0 \\ 0 & 0 & \frac{\beta}{\pi\omega_2 R_2^2} & 0 & 0 \end{bmatrix}. \quad (11)$$

It is easy to verify that the symplex matrix (11) satisfies the Jacoby identity, and the right side of the linearized equations (6), (7) and (9) is a product of the simplex matrix and the Hamiltonian gradient. Both physical meaning of the Hamiltonian (10) and correspondence of the above-deduced equations to the previously constructed theory of a synoptic vortex evolution on the ß-plane become evident after the following transformations of equations (6), (7) and (9). Let us introduce $U = \frac{dX_1}{dt}$, $V = \frac{dY_1}{dt}$. Time differentiation of (6) yields

$$2\pi R_1^2 \frac{dU}{dt} = \pi\omega_2 R_2^2 (\frac{dY_2}{dt} - V) = -\pi(\omega_1 R_1^2 + \omega_2 R_2^2)V\,, \quad (12)$$

$$2\pi R_1^2 \frac{dV}{dt} = -\pi\omega_2 R_2^2 (\frac{dX_2}{dt} - U) =$$

$$= \pi(\omega_1 R_1^2 + \omega_2 R_2^2)U + \pi\omega_2 R_2^2 \frac{\beta}{8}(2R_1^2 - R_2^2) + \pi\omega_1 R_1^2 \frac{\beta R_1^2}{8}. \quad (13)$$

Thus, the following system of equations is intended for obtaining velocity of the trap zone center and its vorticity movement:

$$2\pi R_1^2 \frac{dU}{dt} + \pi(\omega_1 R_1^2 + \omega_2 R_2^2)V = 0\,, \quad (14)$$

$$2\pi R_1^2 \frac{dV}{dt} - \pi(\omega_1 R_1^2 + \omega_2 R_2^2)U + \pi\omega_2 R_2^2 \frac{\beta}{8}(2R_1^2 - R_2^2) + \pi\omega_1 R_1^2 \frac{\beta R_1^2}{8} = 0\,, \quad (15)$$

$$\frac{d\omega_1}{dt} = -\beta V\,. \quad (16)$$

Hamiltonian (10) is also expressed by the variables of the system of equations (14) – (16)

$$H(U, V, \omega_1) = \pi R_1^2 (U^2 + V^2) + \pi\omega_1\omega_2 R_2^2 \frac{1}{8}(2R_1^2 - R_2^2) + \pi\omega_1^2 R_1^2 \frac{R_1^2}{16}. \quad (17)$$

The system of equations (14) – (16) takes on the Hamiltonian form with Hamiltonian (17) in case the following simplex matrix is introduced:

$$G=\begin{bmatrix} 0 & -\dfrac{\pi(\omega_1 R_1^2+\omega_2 R_2^2)}{\left(2\pi R_1^2\right)^2} & 0 \\ \dfrac{\pi(\omega_1 R_1^2+\omega_2 R_2^2)}{(2\pi R_1^2)^2} & 0 & \dfrac{\beta}{\pi R_1^2} \\ 0 & -\dfrac{\beta}{\pi R_1^2} & 0 \end{bmatrix}. \tag{18}$$

Hamiltonian (17) is represented as a sum of three terms. The first and the second ones are equal

$$E_1=E_2=\frac{\pi R_1^2}{2}(U^2+V^2) \tag{19}$$

and represent the vortex translational kinetic energy and the virtual mass energy arising at flowing around the trap zone. The sense of the last term in equation (17) becomes evident if the energy of the liquid particles' rotation within the trap zone is calculated. At that it is sufficient to retain only the basic terms in the expression for the current function having neglected different locations of the vortex core and the trap zone centers. In such an approximation the stream function is as follows:

$$\psi\approx\psi(r)=\begin{cases}\dfrac{1}{4}\omega_1(r^2-R_1^2)+\dfrac{1}{4}\omega_2(r^2-R_2^2),\ r\le R_2,\\ \dfrac{1}{4}\omega_1(r^2-R_1^2)+\dfrac{\omega_2 R_2^2}{2}\ln\left(\dfrac{r}{R_2}\right),\ R_2<r\le R_1.\end{cases} \tag{20}$$

Using expression (20) we obtain

$$E_3=\frac{1}{2}\iint(\nabla\psi)^2 dS=\pi\int_0^{R_1} r\left(\frac{d\psi}{dr}\right)^2 dr=\pi\int_0^{R_1}\frac{1}{4}\omega_1{}^2 r^3 dr+$$
$$+\frac{1}{2}\pi\omega_1\omega_2\int_0^{R_2} r^3 dr+\frac{1}{2}\pi\omega_1\omega_2 R_2^2\int_{R_2}^{R_1} dr+\text{const}(t)=$$
$$=\frac{\pi\omega_1^2 R_1^4}{16}+\frac{\pi\omega_1\omega_2 R_2^4}{8}+\frac{\pi\omega_1\omega_2(R_1^2-R_2^2)}{4}+\text{const}(t), \tag{21}$$

that (taking no account of the time-independent constant in formula (21)) provides the latter term in Hamiltonian (17). Thus, it follows from the general theory [5] that Hamiltonian (17) constitutes the energy.

The above-represented Hamiltonian formulation of the problem of a vortex evolution on the *ß*-plane is constructed for the originally circular vortex with constant vorticity. However, the above interpretation of the Hamiltonian as a sum of the energies of the vortex translation, the liquid particles' rotation within the trap zone and the liquid particles flowing around the trap zone makes it possible to generalize the above-obtained results and to apply them to the vortices which are of rather arbitrary radial-symmetric initial form. The stream function within the trap zone is approximately of the following form

$$\psi=\psi_0(r)+\frac{1}{4}\omega(r^2-R^2), r\le R, \tag{22}$$

where the trap zone radius and its vorticity are denoted as R and ω, respectively, and $\Psi_0(r)$ is the initial stream function. The energy of the liquid particles' rotation within the trap zone is preset by the following expression

$$E_3 = \frac{\pi\omega^2 R^4}{16} + \pi\omega\int_0^R r^2 \frac{d\psi_0}{dr} dr + \text{const}(t), \tag{23}$$

the Hamiltonian representing the vortex energy has the form

$$H = \pi R^2 (U^2 + V^2) + \frac{\pi\omega^2 R^4}{16} + \pi\omega\int_0^R r^2 \frac{d\psi_0}{dr} dr\,, \tag{24}$$

and the simplex matrix is the same as before

$$Q = \begin{bmatrix} 0 & -\frac{\Gamma}{(2\pi R^2)^2} & 0 \\ \frac{\Gamma}{(2\pi R^2)^2} & 0 & \frac{\beta}{2\pi R^2} \\ 0 & -\frac{\beta}{2\pi R^2} & 0 \end{bmatrix}, \tag{25}$$

where circulation on the trap zone boundary is now prescribed by the following expression

$$\Gamma = \pi R^2 \omega + 2\pi R \frac{d\psi_0(R)}{dR}\ . \tag{26}$$

At that the equations describing the vortex evolution takes on the following form

$$2\pi R^2 \frac{dU}{dt} + \Gamma V = 0\,, \tag{27}$$

$$2\pi R^2 \frac{dV}{dt} - \Gamma U + \beta \iint (\psi - \psi_\Gamma) dS = 0\,, \tag{28}$$

$$\frac{d\omega}{dt} = -\beta V\ \ , \tag{29}$$

since it follows from equations (23) used for the rotation energy and (22) – for the stream function that

$$\frac{dE_3}{d\omega} = -\iint (\psi - \psi_\Gamma) dS\ , \tag{30}$$

where ψ_Γ is the stream function value on the trap zone boundary.

Vortex motion and oscillations. Equations (27) and (28) constitute the momentum balance and completely correspond (except for the terms with the time derivative) to the equations describing the vortex motion on the *ß*-plane deduced in [1, 2]. The summands with the time derivative correspond to the momentum change both of the very vortex and the added mass. The terms proportional to the circulation on the trap zone boundary represent the Zhukovsky force influencing the vortex. Finally, the integral in equation (28) is a component of the Rossby force arising due to the *ß*-effect. Relation (30) shows that the Rossby force role consists in converting the rotation energy into the translation one. In contrast to [1, 2], equations (27) and (28) do not include the wave resistance force since radiation of the Rossby waves during the vortex motion is not taken into account in the considered model that, no doubt, excludes regular vortex displacement along the meridian. As a result, the model allows us to describe only the vortex zonal motion and oscillations of the trap zone and the vortex core centers around the equilibrium position. Note that just application of the method in [4] provides substantiation of including the inertial terms in the vortex motion equations used for a particular case of a vortex with a circular core of constant vorticity and permits to generalize this result for the case of a vortex with arbitrary configuration.

The system of equations (27) – (29) has a stationary solution reflecting balance of the Zhukovsky and Rossby forces, and describing the vortex zonal translation. At that the velocity meridian component is equal to zero and the vortex moves to the west. Zonal component of the vortex motion velocity U depends on the trap zone radius R and its vorticity ω. Proceeding from the notions in [1, 2] let us assume that the trap zone boundary is a stream function separatrix in a moving coordinate system associated with the vortex. Then it follows from (3) and (4) that

$$\Gamma = \Gamma_0 = 4\pi Rc\,, \qquad (31)$$

where $c = -U$ is a value of velocity of the vortex uniform motion. In such a case equation (28) and formula (26) connect the vortex uniform motion velocity c and its radius R with the trap zone vorticity ω_0 which is proportional to the vortex meridian translation relative to its initial position.

Let us return to the equations (6), (7) and (9) which permit to interpret the vortex zonal self-propagation in a new way. It is easy to get convinced that stationary movement of the vortex core and the trap zone can be directed only zonally and in such a way that the trap zone and the vortex core centers should be located on one and the same meridian and the trap zone vorticity should not change in time, $\omega_1 = \omega_1^0$. As a result, the equations (6), (7) and (9) can be reduced to two equations

$$\begin{aligned} -c &= \frac{\omega_2 R_2^2}{2R_1^2}(Y_2 - Y_1) - \frac{\beta R_1^2}{8}, \\ -c &= -\frac{1}{2}\omega_1 (Y_2 - Y_1) - \frac{\beta R_1^2}{4} + \frac{\beta R_2^2}{8}. \end{aligned} \qquad (32)$$

Having excluded the vortex motion velocity c from equation (32) we find that

$$\left(\omega_1 R_1^2 + \omega_2 R_2^2\right)(Y_2 - Y_1) + \frac{\beta R_1^2}{4} - \frac{\beta R_2^2}{4} = 0$$

and, thus, the vortex core center is shifted to the south relative to the trap zone center. Due to this circumstance and the ß-effect contribution, the vortex self-propagation to the west occurs.

Now let us consider the characteristics of the vortex core and the trap zone centers' oscillations. First, note that, owning to (26) and (31), the circulation Γ included in equations (27) and (28) can be represented as a function of the trap zone vorticity:

$$\Gamma = \Gamma_0 + \pi R^2 \left(\omega - \omega_0\right). \qquad (33)$$

In view of (23), the condition of the vortex uniform motion following from equation (28) similarly yields

$$\frac{dE_3}{d\omega} = -\iint(\psi - \psi_\Gamma)dS = \frac{\pi(\omega - \omega_0)R^4}{8} - \frac{c\Gamma_0}{\beta}. \qquad (34)$$

Note that the vortex radius in all the above formulas does not change in time since the oscillation frequency is assumed to be much higher than the characteristic time of the vortex evolution.

In view of (33) and (34), equations (27) and (28) are as follows

$$2\pi R^2 \frac{dU}{dt} + \left(\Gamma_0 + \pi R^2(\omega - \omega_0)\right)V = 0, \qquad (35)$$

$$2\pi R^2 \frac{dV}{dt} - \left(\Gamma_0 + \pi R^2(\omega - \omega_0)\right)U + c\Gamma_0 - \beta\frac{\pi(\omega - \omega_0)R^4}{8} = 0. \qquad (36)$$

Having used (29) for making a closed system of equations, and combining (29) and (35) we find

$$U = -c + \frac{2c}{\beta R}(\omega - \omega_0) + \frac{(\omega - \omega_0)^2}{4\beta}. \qquad (37)$$

Substituting (37) into (35) and using (29) once again, we obtain the nonlinear equation (for description of oscillations) which can be integrated in the elliptic functions. Note that frequency of small oscillations of the vortex motion velocities σ is derived from the linearized equation:

$$\sigma^2 = 4\frac{c^2}{R^2} - \frac{\beta c}{2} + \left(\frac{\beta R}{4}\right)^2. \qquad (38)$$

It would be interesting to compare the expression obtained for frequency of the vortex and the trap zone centers' oscillations with the results of numerical simulations.

Conclusion. Application of the "elliptic approach" [4] to studying evolution of a single synoptic vortex on the *ß*-plane has permitted to generalize the earlier-proposed theory by supplementing balance of the forces influencing the vortex with the inertial summands. It is shown that the deduced evolution equations are of the Hamiltonian form in case the non-canonical character of the used set of variables is taken into account. The evolution equations describing the vortex core and the trap zone center motion for a particular case of the initially circular vortex with constant vorticity, are written in two ways: the "Lagrangian" form (the variables are the trap zone vorticity, and the coordinates of the vortex core and the trap zone center) and the "Euler" one (the variables are the vortex motion velocity components and the trap zone vorticity). Analysis of the "Lagrangian" solution permits to interpret the vortex self-propagation on the *ß*-plane without basing on the force balance. It is revealed that the vortex core and the trap zone centers are displaced along the meridian; and their synchronous movement is similar to translation of a pair of point vortices with identical intensity but different signs.

Being supplemented with the inertial terms, the vortex evolution equations provide a possibility of occurring of high-frequency oscillations of the vortex motion velocity actually observed in numerical simulations. The represented theory allows one to calculate frequency of such oscillations.

In the present study, the process of the Rossby waves' radiation by a moving vortex was not deliberately taken into consideration. If one describes only the vortex smooth evolution at which the inertial terms are negligible, the effect of the Rossby wave radiation can be taken into account by including the wave resistance force calculated in [1, 2] to the total force balance. However, oscillations of the trap zone center can change the radiation conditions, therefore a more complete problem requires additional research.

Finally, note that actually a “circular approach” (but not an “elliptical one”) was applied for the form of the vortex core and the trap zone. In spite of bulky computations, it seems interesting to consider a more flexible elliptical approach which can assess the vortex stability.

The work is performed within the framework of the state task № 0827-2014-0011 "Research of regularities of the marine environment changes based on the operational observations and the data obtained from the system of nowcast, forecast and reanalysis of the marine water areas’ state” (code “Operational oceanography”).

REFERENCES

1. Korotaev, G.K., 1980, “*Asimptotichesky rezhim dinamiki intensivnogo barotropnogo vikhrya* [Asymptotic mode of the intensive barotropic eddy dynamics]”, *Morskoy gidrofizichesky zhurnal*, no. 1 (88), pp. 5-18 (in Russian).
2. Korotaev, G., Fedotov, A., 1994, “Dynamics of an isolated barotropic eddy on a beta-plane”, *J. Fluid Mech*., vol. 264, pp. 277-301.
3. Lam, J.S.-L., Dritschel, D.G., 2001, “On the beta-drift on an initially circular vortex patch”, *J. Fluid Mech*., vol. 436, pp. 107-129.
4. Legras, B., Dritschel, D.G., 1991, “The elliptic model of two-dimensional vortex dynamics. I: The basic state”, *Phys. Fluid A*, vol. 3, no. 5, pp. 845-854.
5. Shepherd, T.G., 1990, “Symmetries, conservation laws, and Hamiltonian structures in geophysical fluid dynamics”, *Adv. Geophys*., vol. 32, pp. 287-338.