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published in MORSKOY GIDROFIZICHESKIY ZHURNAL, Vol. 39, Iss. 3 (2023)

Original article

Phase Shifts in the Counter-Interaction of Shallow Water Waves

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Abstract
Purpose. The work is aimed at numerical studying and describing the wave effects arising from the counter-interaction of different polarity single pulses within the framework of the Boussinesq type equation system with regard to dispersion in a constant depth basin.
Methods and Results. To simulate the scenarios of the long wave pulse interaction, the *CLAWPACK* software package was used. It included the hybrid method for numerical solving the equation system which, in its turn, comprised the method of finite volumes and finite differences. The results were compared to the numerical solutions obtained earlier by using a non-dispersive nonlinear system of shallow water equations.
Conclusions. The fundamental wave phase shift is studied in its interaction with the counter-propagating pulses of different polarity. It is shown that the phase shift increases with the initial pulse amplitude growth. The dispersion influence is manifested in a single wave transformation into an undular bore. The study novelty consists in detecting and demonstrating such nonlinear effects as the phase shifts in the long wave counter-interaction within the framework of the nonlinear shallow water numerical model including the dispersion terms.

Keywords: long waves, numerical experiment, Boussinesq equations, wave interaction

Acknowledgments: The represented results were obtained with the support of the RSF grant 22-17-00153.

For citation: Rodin, A.A., Rodina, N.A., Trusova, A.Yu. and Pelinovsky, E.N., 2023. Phase Shifts in the Counter-Interaction of Shallow Water Waves. *Physical Oceanography*, 30(3), pp. 265-273. doi:10.29039/1573-160X-2023-3-265-273

DOI: 10.29039/1573-160X-2023-3-265-273

Introduction

Catastrophic waves (tsunamis and storm surges) are long waves, thus long-wave models are widely used to study their propagation. Owing to the use of conservation laws that allow shock fronts to be taken into account, the hyperbolic system of nonlinear shallow water equations has become a standard mathematical model for simulation of long tsunami wave propagation and runup in coastal areas [1–5].

The theory of shallow water does not consider the dispersion of waves, which, nevertheless, turns out to be significant when long waves of the tsunami type propagate over long distances [6–8]. It is taken into account in the framework of the so-called Boussinesq equations, in particular, the Green – Naghdi system [9, 10]. It should be noted that the Boussinesq equations are not derived exactly from the original Euler equations. There are many types of Boussinesq equations [11–13] that have been used to solve specific problems in the field of marine hydraulic engineering and coastal oceanography [8, 14, 15].

The counter interaction of waves in shallow water was studied within the framework of the "pure" theory of shallow water [16, 17] and Boussinesq equations [18, 19]. It was shown that, compared with the linear theory, the nonlinearity in the wave height leads to a greater increase at the moment of collision, especially in the interaction of solitons having limiting amplitude. Phase shift effects in the interaction of nonlinear waves were not considered in this case. The present paper is devoted to the numerical study and description of wave effects arising from the counter-interaction of different polarity single pulses within the framework of the Boussinesq type equation system with regard to dispersion in a constant depth basin. Numerical calculations were carried out using the BOUSSCLAW computer system for Boussinesq equations [20].

Mathematical model

The Boussinesq system in the form proposed in [20] was chosen as the initial equations for the wave counter-interaction analysis:

$$H_t + (Hu)_x = 0, \tag{1}$$

$$(1 - D)[(Hu)_t] + (Hu^2 + \frac{g}{2}H^2)_x - gHh_x - Bgh^2(h\eta_x)_{xx} = 0. \tag{2}$$

Here $H(x, t) = h(x) + \eta(x, t)$ is the total water depth; $\eta(x, t)$ is the water surface shift; $h(x)$ is the undisturbed water depth; $u(x, t)$ is the depth-averaged horizontal flow velocity; g is the gravitational constant. The D operator for any $w(x, t)$ is defined using the auxiliary variable $w(x, t)$ as follows:

$$D(w) = \left(B + \frac{1}{2}\right)h^2 w_{xx} - \frac{1}{6}h^3 (\frac{w}{h})_{xx}. \tag{3}$$

The dispersion B parameter is chosen to be 1/15 [20], while the linear dispersion relation value, which follows from the Boussinesq equations, better corresponds to the exact relation for water waves. At $B = 0$ and $D = 0$, the system of equations (1) – (2) transforms into the known nonlinear system of shallow water.

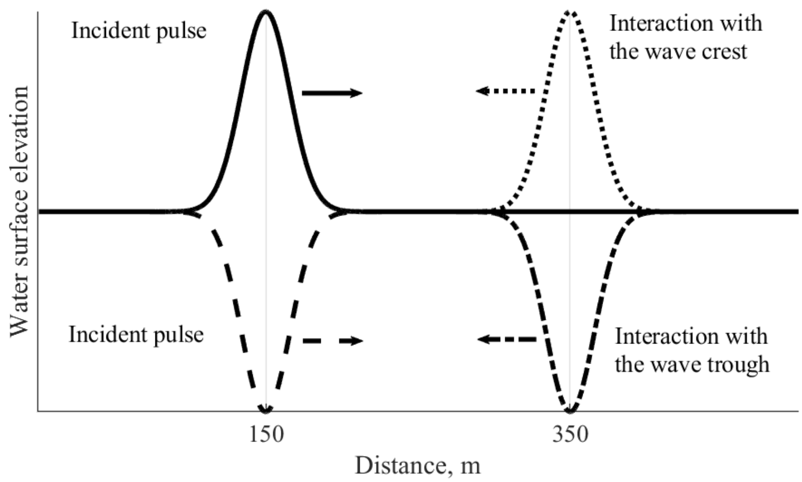
All numerical experiments were carried out using the CLAWPACK software, more precisely, its BOUSSCLAW add-on (www.clawpack.org). The BOUSSCLAW computing package uses a hybrid method for the numerical solution of the system of equations (1) – (2), including the method of finite volumes and finite differences. In particular, the finite volume method is used for the non-linear part of the equations and finite difference discretization with fractional steps for additional terms, such as the standard and higher order variance terms.

In the present study, a pool of constant depth h = 1 m and a length of 1000 m was used. The boundary conditions at the computational domain ends were the free drift conditions formulated strictly without taking into account dispersion, but in fact, the numerical calculation stopped until the moment when the wave approached the computational domain edge. The spatial step value was 0.16 m, the time step was selected automatically, taking into account the Courant–Friedrichs–Levy stability criterion [21].

At the initial moment of time, one or two Gaussian pulses were given

$$H(x,0) = h + A\exp[-\beta(x-x_1)^2] + A_2\exp[-\beta(x-x_2)^2] \quad (4)$$

having length ~ 120 m (β = 0.002 m^{-2}) near the sole. The pulses are separated in space (x_1 = 150 m, x_2 = 350 m). The amplitudes of A and A_2 pulses are assumed to be identical in module but may differ in signs (Fig. 1). This makes it possible to study the interaction of waves in the form of a crest and in the form of a trough, while the nonlinear effects for them are manifested in different ways. The flow velocity at the initial moment of time is equal to zero, so the initial impulse is divided into two symmetrical waves running in opposite directions and having amplitudes twice as small as the initial one. The initial amplitudes of the A and A_2 waves vary from 0.1 m (weak non-linearity) to 1 m (strong non-linearity).

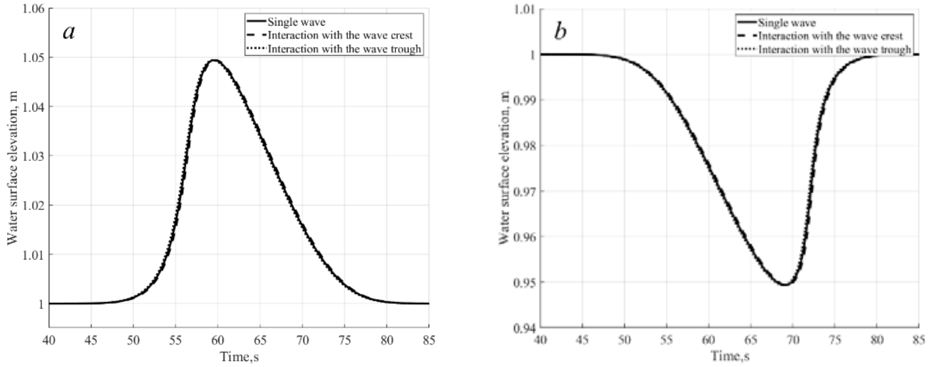


F i g. 1. Initial form and location of pulses

Interaction of small amplitude waves

At a small wave amplitude (0.1 m), each pulse breaks up into two half-amplitude pulses, as follows from the linear theory. The nonlinear effects are very weak, but they lead to a wave profile distortion over a long time (Fig. 2).

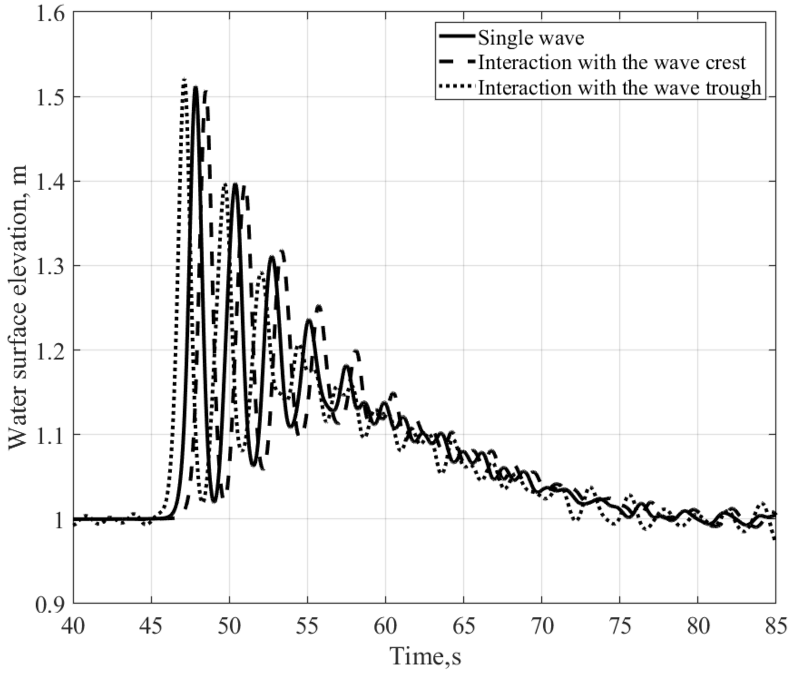
Dispersion has almost no effect at such distances, since a sufficiently long wave is given. The phase shifts are weak enough that they can be ignored. Meanwhile, the dynamics of the positive and negative impulses occur in different ways already within the small amplitude limits. As is known [22], the nonlinearity is greater in the trough (where the total depth is smaller) than on the crest. Therefore, the wave crest was deformed weaker than the base (Fig. 2, *a*), where dispersion-induced oscillations are already beginning to appear.



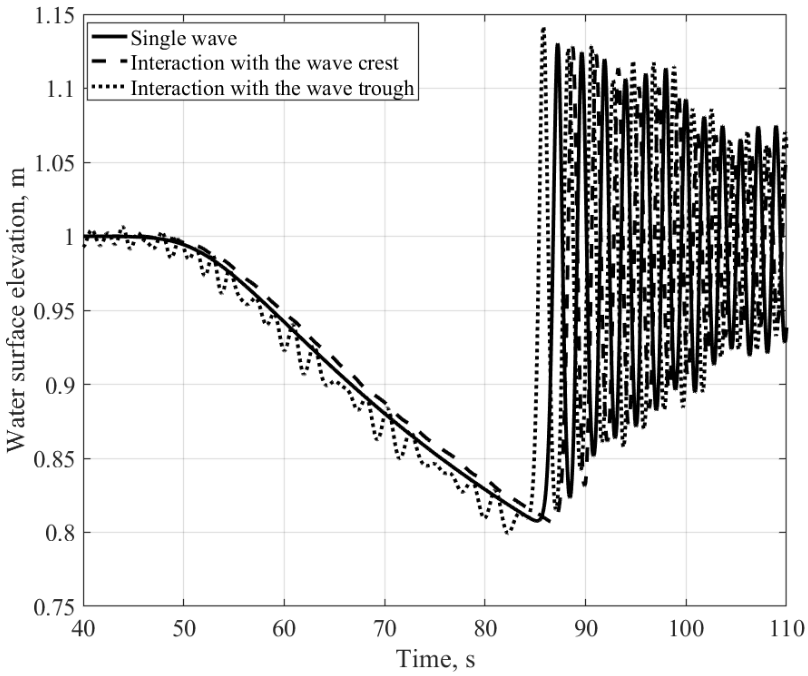
F i g. 2. The wave oscillogram at point x = 300 m with the initial amplitude 0.1 m at the positive (*a*) and negative (*b*) polarity

Interaction of moderate amplitude waves

In this experiment, the amplitude of the initial pulses was 0.6 m, i.e., the interaction took place between two incident pulses with an amplitude half that of the initial one. If dispersion is not taken into account (i.e., we work within the framework of the nonlinear theory of shallow water), then the nonlinearity will lead to the shock wave formation and its amplitude will decrease. In this case, the interaction of shock waves occurs inelastically, and phase shifts become noticeable, which “slow down” the incident pulse when interacting with the crest and “accelerate” it when interacting with the trough.



F i g. 3. The wave oscillogram of the positive polarity at point x = 300 m with the initial amplitude 0.6 m



F i g. 4. The wave oscillogram of the negative polarity at point x = 300 m with the initial amplitude 0.6 m

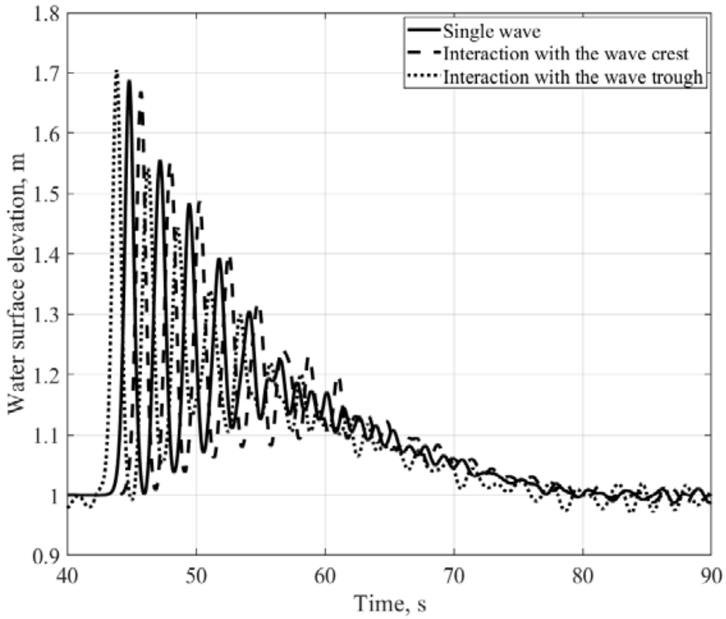
All these effects we studied earlier in the dispersionless model framework [22]. The dispersion model shows new effects, consisting in the appearance of large oscillations on the main wave body, when the so-called undular bore is formed (Fig. 3, 4) [23, 24].

Fig. 3 shows interaction scenarios of an incident positive polarity wave with counterpropagating waves. Here, the main wave amplitudes at the point x = 300 m slightly exceeded 0.5 m (0.53 m for the scenario of interaction with a trough, 0.52 m for a scenario without interaction, 0.51 m for a scenario with a crest). The difference in amplitude here is caused by the presence of non-linear oscillations remaining in the counter wave tail. The phase shift in Fig. 3 is 0.7 sec.

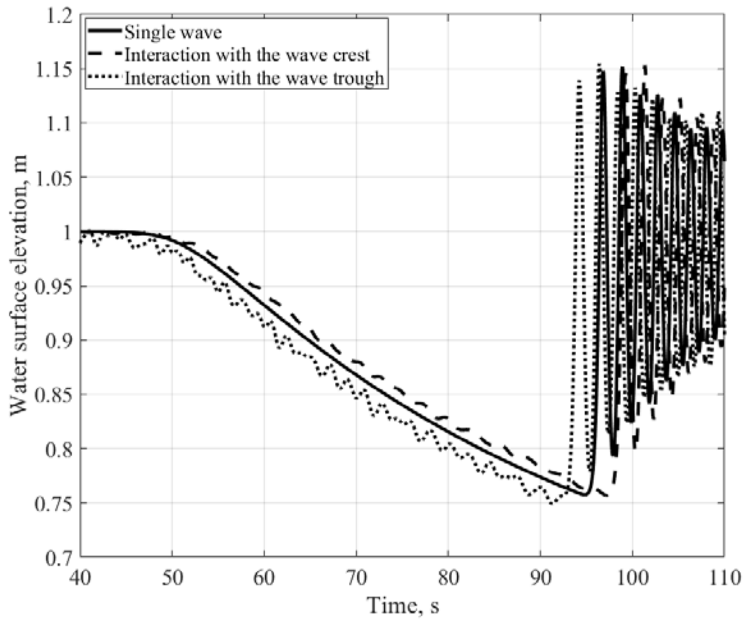
Fig. 4 shows three interaction scenarios of an incident negative polarity wave with counterpropagating waves. Here, as in the previous case, a small difference in the amplitudes is observed and, at the same time, oscillations in the opposite trough tail are more clearly manifested (Fig. 4, dotted line up to the time of 80 sec). In this case, the phase shift between the scenarios is 1.5 sec, and the trough amplitude in all three scenarios is ~ 0.19 m.

Interaction of high amplitude waves

In this case (the initial pulse amplitude is 0.9 m), the nonlinear and dispersion effects become dominant and are already noticeable at the beginning of the numerical experiment. Here, the process of wave decay into solitons goes much faster (Fig. 5, 6).



F i g. 5. The wave oscillogram positive polarity at point x = 300 m with the initial amplitude 0.9 m



F i g. 6. The wave oscillogram of the negative polarity at point x = 300 m with the initial amplitude 0.9 m

As for phase shifts during the interaction of waves, they turn out to be qualitatively the same as in the previously considered scenarios. It is noted that, as in the previous case, with a negative polarity, the dispersion tail "stretches" behind the pulse, which turns out to be ahead of the head wave front, which is observed at the point x = 300 m before the incident wave comes (Fig. 6). Changes in the amplitudes of the second and subsequent solitons are also visible. In the scenario of interaction with the crest, the amplitudes of the solitons following the main pulse are greater than in the scenario without interaction (Fig. 5), while in the case of interaction with the trough, on the contrary, they are smaller (Fig. 6). The phase shift for such scenarios was 1 sec compared to the scenario without interaction.

When interacting with the impulse of an incident negative polarity wave (Fig. 6), the influence of the nonlinear dispersion tail following the trough is reflected in the tide gauge record by this wave arrival moment (40 sec from the beginning of the experiment). The amplitudes of these pulses are ~ 0.25 m, while the phase shift is 2.2 sec.

Conclusion

In the present paper, six different scenarios for the propagation and counter-interaction of different polarity pulses in the framework of Boussinesq-type equations with allowance for dispersion effects are considered. The results are given for three different initial impulse amplitudes: 0.1, 0.6 and 0.9 m.

In the first two scenarios, a single impulse is set. It eventually breaks up into two symmetrical waves running in opposite directions; in the third and fourth scenarios, the incident pulse interacts with a wave of positive polarity (crest) running in the opposite direction and formed as a result of the decay of an identical shape and amplitude pulse; in the fifth and sixth scenarios, a similar interaction with the wave trough is carried out.

The interaction of a falling small amplitude crest with different polarity pulses is accompanied by small phase shifts (0.2 sec). The interaction of the incident negative polarity wave with the crest and trough in this case, due to the greater nonlinearity, leads to the appearance of dispersion effects observed in the main wave tail. The phase shifts are also small here.

As the amplitude increases, the effects of dispersion become stronger, especially for an incident negative polarity wave. Phase shifts also increase. Thus, for incident waves with an amplitude of 0.6 m and positive polarity, the phase shift is 0.7 sec compared to the scenario without interaction, and with an initial amplitude of 0.9 m it is 1 sec.

The strongest nonlinear interaction is manifested for negative polarity waves, since the ratio of the wave amplitude to the local depth increases. In the scenario with an incident negative polarity wave, with an initial pulse amplitude of 0.5 m, the phase shifts are 1.5 sec compared to the interaction scenario, and with an amplitude of 0.9 m, they are 2.2 sec.

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Natalya A. Rodina – visualization of the results of numerical modelling

Anna Yu. Trusova – help with GeoClaw documentation and presentation of results

Efim N. Pelinovsky – formulation of the problem, theoretical and analytical calculations, interpretation of the results of numerical calculation

The authors have read and approved the final manuscript.
The authors declare that they have no conflict of interest.